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Alicki's model of scattering-induced decoherence derived from Hamiltonian dynamics

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Abstract

We study a semiphenomenological model introduced by Alicki (2002 *Phys. Rev.* A **65** 034104), describing environmental decoherence by scattering of a Brownian particle in a gas environment. For a slightly wider class of models, we prove that the semigroup describing the dynamics of the Brownian particle can be approximated by the reduced dynamics arising from a Hamiltonian interaction between the particle and an infinite fermionic thermal gas reservoir, provided the scattering process is isotropic.

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1. Introduction

In the past 20 years environmental decoherence has been the subject of intensive theoretical and by now also experimental research [1, 2]. It addresses the question why the objects surrounding us obey the laws of classical physics, despite the fact that our most fundamental physical theory, quantum theory, when directly applied to these objects, results in contradictions to what is observed. This is an embarrassing situation, since on the other hand, quantum theory has seen a remarkable success and an ever increasing range of applicability. Thus, the question how to reconcile quantum theory with classical physics is a fundamental one, and efforts to find answers to it persisted from the inception of quantum theory in the 1920s until today. Although there is still no general consensus how an answer can be achieved, environmental decoherence is the most promising one and the one most widely discussed.

The answer that the programme of environmental decoherence gives to the question posed above is that quantum theory is also valid in the macroscopic domain, but that one has to take into account the fact that macroscopic objects are usually strongly interacting with their environment, which ultimately leads to a classical behaviour of the system. Thus, classicality is a dynamically emergent phenomenon due to the interaction of quantum systems with other quantum systems surrounding them.

Of particular interest is the question why macroscopic objects always appear localized, although the most fundamental principle of quantum theory, the superposition principle, allows macroscopic objects to be in states with no well-defined position. In the simplest situation, the macroscopic object considered is a massive Brownian particle with no further internal structure which undergoes scattering with particles of a thermal gas environment; the observable which becomes classical in this case is its centre-of-mass position. This situation has been experimentally realized in matter wave interferometry with fullerene molecules, allowing the observation of progressive decoherence which is due to scattering with background gases [3]. A number of models have been devised to describe this situation [4–6]. They employ scattering theory and some simplifying mathematical and physical assumptions, valid for a certain range of parameters, to derive a Markovian reduced dynamics for the Brownian particle. Another approach to the problem of obtaining a reduced Markovian dynamics is by considering a Hamiltonian dynamics for the total system and doing a perturbative calculation, i.e., performing a Markovian limit as is done at an abstract level in the general theory of Markovian limits [7]. Such a derivation has been given in [6], however, not in a mathematically rigorous fashion with an explicit proof of convergence of the reduced dynamics to a Markovian one.

In this paper we prove, using a singular coupling limit, that the semiphenomenological model describing localization of a Brownian particle given by Alicki [8] can be approximated by the reduced dynamics arising from a Hamiltonian interaction between the particle and an infinite fermionic thermal gas reservoir, the interaction being linear in the field operators of the gas. We choose a fermionic gas because it admits bounded field operators and thus avoids mathematical complications in dealing with unbounded operators, e.g., the self-adjointness of the Hamiltonian and in the proof of theorem 3. The proof is performed for a slightly wider class of models which contains Alicki's as a member. Such a derivation is important since it justifies the assumption of a Markovian time evolution of the Brownian particle, which is used in many applications, e.g., to interpret experimental data. Moreover, by taking into account the approximations made it may serve to identify the range of validity of Markovian approximations.

The paper is organized as follows: in section 2 we present the general mathematical framework and the class of models we shall work with. Section 3 gathers the necessary results from the theory of Markovian limits. In section 4 we set up a Hamiltonian model and prove that its reduced dynamics becomes Markovian in a singular coupling limit, and in section 5 we use this result to show that the dynamics described by a certain class of semigroups can be approximated by the reduced dynamics of a Hamiltonian interaction.

2. General framework

Let G be a locally compact Hausdorff topological group with unit e, and $\mathscr{B}_0(G)$ the σ-algebra of Baire sets in G. Let $\mathscr{M}(G)$ be the set of all Baire measures, i.e., of all measures μ on the measurable space $(G, \mathscr{B}_0(G))$ for which $\mu(K) < \infty$ for all compact subsets $K \subseteq G$. Let $\mathscr{M}^{\mathrm{b}}(G)$ denote the set of all bounded measures, $\mathscr{M}^1(G)$ the set of all normalized (probability) measures, and $\mathscr{M}_+(G)$ the set of all nonnegative measures in $\mathscr{M}(G)$. The set of real-valued continuous functions on G with compact support is denoted by $\mathscr{K}(G)$, we define for $\mu \in \mathscr{M}(G)$ and $f \in \mathscr{K}(G)$ a dual pairing by $\langle \mu, f\rangle = \int_G f \,\mathrm{d}\mu$. The weak topology on $\mathscr{M}(G)$ with respect to this pairing is called the vague topology. Let $\{\mu_t\}_{t\geqslant 0} \subseteq \mathscr{M}^1(G)$ be a one-parameter family of Baire measures. It is called a one-parameter convolution semigroup of Baire measures if $\mu_t * \mu_s = \mu_{t+s}$ for all $t, s \geqslant 0$ where $*$ denotes convolution,

$[0, \infty[\ni t \mapsto \mu_t$ is continuous in the vague topology on $\mathscr{M}(G)$, and $\mu_0 = \delta_e$, where δ_g denotes the Dirac measure of unit mass concentrated in $g \in G$.

Let $\mathscr{H}$ be a Hilbert space and denote by $\mathscr{L}(\mathscr{H})$ the C*-algebra of all bounded operators, by $\mathsf{U}(\mathscr{H})$ the set of all unitary operators and by $\mathsf{T}(\mathscr{H})$ the Banach space of all trace class operators on $\mathscr{H}$, endowed with the trace norm $\|\cdot\|_1$. Let $\{T_t\}_{t\geqslant 0}$ be a family of bounded operators on $\mathsf{T}(\mathscr{H})$. It is called a strongly continuous semigroup if $[0, \infty[\ni t \mapsto T_t(\rho) \in \mathsf{T}(\mathscr{H})$ is continuous for all $\rho \in \mathsf{T}(\mathscr{H})$ and if the semigroup property holds: $T_0 = 1$ and $T_t T_s = T_{t+s}$ for all $t, s \geqslant 0$. A strongly continuous semigroup $\{T_t\}_{t\geqslant 0}$ on $\mathsf{T}(\mathscr{H})$ is called a quantum dynamical semigroup if it is completely positive and trace preserving, i.e., $\mathrm{tr}[T_t(\rho)] = \mathrm{tr}[\rho]$ for all $\rho \in \mathsf{T}(\mathscr{H})$ and $t \geqslant 0$. The time evolution of an open quantum system is described by a quantum dynamical semigroup if it is Markovian, i.e., memory free.

In the following, we fix a strongly continuous unitary representation $G \ni g \mapsto U(g) \in \mathsf{U}(\mathscr{H})$, i.e., the map $g \mapsto U(g)\psi$ is continuous for all $\psi \in \mathscr{H}$, and $U(gh) = U(g)U(h)$ for all $g, h \in G$. Then one can prove the following theorem [9].

Theorem 1. *Let $\{\mu_t\}_{t\geqslant 0} \subseteq \mathscr{M}^1(G)$ be a one-parameter convolution semigroup of Baire probability measures. If we define the operators T_t by*

$$\mathsf{T}(\mathscr{H}) \ni \rho \mapsto T_t(\rho) = \int_G U(g)\rho U(g^{-1})\,\mathrm{d}\mu_t(g), \qquad t \geqslant 0, \tag{1}$$

then $\{T_t\}_{t\geqslant 0}$ is a quantum dynamical semigroup on $\mathsf{T}(\mathscr{H})$.

Let $\mu \in \mathscr{M}^1(G)$ and $\alpha > 0$. Consider the family $\{\mu_t\}_{t\geqslant 0}$ of Baire probability measures defined by

$$\mu_t = \mathrm{e}^{-\alpha t} \sum_{n=0}^{\infty} \frac{(\alpha t)^n}{n!} \mu^{*n}, \qquad t \geqslant 0, \tag{2}$$

where $\mu^{*n} = \mu * \cdots * \mu$ (n times) and $\mu^0 = \delta_e$. The sum converges in the vague topology and defines a one-parameter convolution semigroup [9], called the Poisson process on G. Let $\{T_t\}_{t\geqslant 0}$ be the associated quantum dynamical semigroup given by (1). It is easy to show that it is of the form $T_t(\rho) = \mathrm{e}^{L_{\mathrm{D}} t}\rho$ with the bounded generator

$$L_{\mathrm{D}}(\rho) = \alpha \int_G (U(g)\rho U(g^{-1}) - \rho)\,\mathrm{d}\mu(g), \qquad \rho \in \mathsf{T}(\mathscr{H}). \tag{3}$$

Semigroups with generators of the form (3) will be the subject of this paper. In particular, if we take $\mathscr{H} = \mathsf{L}^2(\mathbb{R}^3)$ and if we choose G as the additive group of $\mathbb{R}^3$, the representation U as $U(k) = \mathrm{e}^{-\mathrm{i}k\hat{x}}$ where $k \in \mathbb{R}^3$ and $\hat{x}$ is the position operator on $\mathscr{H}$, and the measure μ as $n\lambda$, which is the Lebesgue measure multiplied by a positive, continuous and integrable weight function $n : \mathbb{R}^3 \longrightarrow \mathbb{R}_{>0}$ such that $\int_{\mathbb{R}^3} n(k)\,\mathrm{d}k = 1$, then the generator (3) takes the form

$$L_{\mathrm{D}}(\rho) = \alpha \int_{\mathbb{R}^3} n(k)(\mathrm{e}^{-\mathrm{i}k\hat{x}}\rho\,\mathrm{e}^{\mathrm{i}k\hat{x}} - \rho)\,\mathrm{d}k, \tag{4}$$

which is the dissipative part of the generator of Alicki's semiphenomenological model [8]. The total generator is then given by $L(\rho) = -\mathrm{i}[H, \rho] + L_{\mathrm{D}}(\rho)$, where $\rho \in \mathrm{dom}[H, \cdot] \subseteq \mathsf{T}(\mathscr{H})$ and $H = \hat{p}^2/2m + V(\hat{x})$ is the Hamiltonian of the Brownian particle in a potential V.

3. Markovian limits

If we consider the reduced dynamics of a quantum open system S interacting with its environment E, we will find in general a non-Markovian behaviour of the system due to the presence of memory effects. However, in certain cases the reduced dynamics obtained by tracing over the degrees of freedom of E may be approximated by a Markovian one in such a way that the approximation becomes exact in a certain limit. This approximation may be achieved by introducing an appropriate scaling parameter λ in the total Hamiltonian H_λ of the joint quantum system with Hilbert space $\mathscr{H} = \mathscr{H}_S \otimes \mathscr{H}_E$, and showing that the reduced dynamics tends to a Markovian one (in an appropriate sense) if we take $\lambda \to 0$. We will employ the singular coupling limit [10] and use the abstract framework of Markovian limits introduced by Davies [11], see also [7].

Let X be a Banach space and P_0 a projection on X with $\|P_0\| = 1$, we put $P_1 = 1 - P_0$ and define $\mathsf{X}_0 = P_0\mathsf{X}$ and $\mathsf{X}_1 = P_1\mathsf{X}$, so that $\mathsf{X} = \mathsf{X}_0 \oplus \mathsf{X}_1$. Suppose that we are given a strongly continuous isometric one-parameter group $\{U_t\}_{t\in\mathbb{R}}$ on X with generator Z and assume that $[U_t, P_0] = 0$ for all $t \in \mathbb{R}$. We define $Z_i = P_i Z = Z P_i$ with domain dom Z_i = dom Z for $i = 0, 1$. Consequently we have $Z = Z_0 + Z_1$. Let A be a bounded operator on X and put $A_{ij} = P_i A P_j$ for $i, j = 0, 1$, we will assume that $A_{00} = 0$ throughout. We now introduce the scaling appropriate for the singular coupling limit: let $\lambda > 0$ and define the operator

$$Z_\lambda = Z_0 + \lambda^{-2} Z_1 + \lambda^{-1} A. \tag{5}$$

Note that since A is bounded, by the bounded perturbation theorem Z_λ is the generator of a group $\left\{U_t^\lambda\right\}_{t\in\mathbb{R}}$ of isometries on X, which describes in physical applications the dynamics of the joint system; the reduced dynamics on X_0 is given by $\left\{P_0 U_t^\lambda P_0\right\}_{t\in\mathbb{R}}$, which is, in general, not Markovian, i.e., not a semigroup. If one defines the operator $K(\lambda, t)$ on X by

$$K(\lambda, t)x = \int_0^{\lambda^{-2}t} \mathrm{e}^{-\lambda^2 Z_0 s} A_{01} \exp((Z_1 + \lambda A_{11})s) A_{10} x \, \mathrm{d}s, \qquad x \in \mathsf{X}, \tag{6}$$

one can prove the following approximation theorem, which is the basis for all further developments.

Theorem 2. *Suppose that for every $t_0 \geqslant 0$ there is a constant $C \geqslant 0$, such that if $|\lambda| < 1$ we have*

$$\sup_{0\leqslant t\leqslant t_0} \|K(\lambda, t)\| \leqslant C. \tag{7}$$

Furthermore, suppose there exists a bounded operator $K \in \mathscr{L}(\mathsf{X}_0)$ such that for all $x \in \mathsf{X}_0$ we have

$$\lim_{\lambda\to 0} \sup_{0\leqslant t\leqslant t_0} \|(K(\lambda, t) - K)x\| = 0. \tag{8}$$

Then it follows that

$$\lim_{\lambda\to 0} \sup_{0\leqslant t\leqslant t_0} \|(P_0 \exp((Z_0 + \lambda^{-2} Z_1 + \lambda^{-1} A)t) P_0 - \exp((Z_0 + K)t))x\| = 0 \tag{9}$$

for all $x \in \mathsf{X}_0$.

For a proof of this theorem see [12]. It is similar to theorem 5.18 of [13], but with the difference that it admits the weaker assumption (8), which can be verified in the singular coupling limit even if Z_0 is unbounded (see also [10]). The operator K in the preceding theorem turns out to be given by

$$K(x) = \int_0^\infty A_{01} \mathrm{e}^{Z_1 s} A_{10} x \, \mathrm{d}s = \int_0^\infty P_0 A \mathrm{e}^{Z_1 s} A P_0 x \, \mathrm{d}s, \qquad x \in \mathsf{X}_0. \tag{10}$$

Note that the limit semigroup $\{\exp((Z_0 + K)t)\}_{t\geqslant 0}$ on X_0 in (9) is indeed a semigroup by the bounded perturbation theorem. Moreover, if X is a Banach algebra, $\exp((Z_0 + K)t)$ is positive, since by (9) it is the strong limit of the positive reduced dynamics.

4. Discrete model

We will now put the abstract framework of sections 2 and 3 to work and introduce a Hamiltonian model which yields in the singular coupling limit a reduced dynamics whose dissipative part is a discrete version of (3). We will use the notation of the previous sections.

Let $\mathscr{H}_\mathrm{S}$ be the Hilbert space of the Brownian particle with Hamiltonian H_S, its configuration space is a locally compact Hausdorff topological group G. Its environment consists of an infinite thermal gas of noninteracting fermions. We employ the algebraic framework of quantum statistical mechanics [14] in which this system is described by the algebra of the canonical anticommutation relations $\mathfrak{A}(\mathfrak{h})$ over the one-particle Hilbert space $\mathfrak{h} = \mathsf{L}^2(\mathbb{R}^3)$, generated by the bounded creation and annihilation operators $a_\mathrm{F}^*(f), a_\mathrm{F}(f), f \in \mathfrak{h}$, e.g., those of the Fock representation, which obey the canonical anticommutation relations. Since the particles are noninteracting, their one-particle Hamiltonian is given by $H = -\Delta$ on $\mathfrak{h}$, inducing a time evolution given by a strongly continuous one-parameter group of Bogoliubov transformations $\{\tau_t\}_{t\in\mathbb{R}}$ on $\mathfrak{A}(\mathfrak{h})$, defined by $\tau_t(a_\mathrm{F}(f)) = a_\mathrm{F}(\mathrm{e}^{\mathrm{i}tH} f)$. The thermal equilibrium state is the unique KMS state ω with respect to $\{\tau_t\}_{t\in\mathbb{R}}$ and inverse temperature $\beta > 0$, which is quasi free and gauge invariant, and uniquely determined by its two-point function

$$\omega(a_\mathrm{F}^*(f)a_\mathrm{F}(g)) = \langle g|\mathrm{e}^{-\beta H}(1+\mathrm{e}^{-\beta H})^{-1} f\rangle = \int_{\mathbb{R}^3} \overline{\hat{g}(p)}\hat{f}(p)\frac{\mathrm{e}^{-\beta p^2}}{1+\mathrm{e}^{-\beta p^2}}\,\mathrm{d}p, \tag{11}$$

with $f, g \in \mathfrak{h}$, and $\hat{f}, \hat{g}$ denote the Fourier transforms of f and g. Let $\pi : \mathfrak{A}(\mathfrak{h}) \longrightarrow \mathscr{L}(\mathscr{H}_\mathrm{E})$ be the GNS representation [14] of $\mathfrak{A}(\mathfrak{h})$ on $\mathscr{H}_\mathrm{E}$ with respect to ω with cyclic vector $\Omega \in \mathscr{H}_\mathrm{E}$, i.e., $\omega(x) = \langle\Omega|\pi(x)\Omega\rangle$ for all $x \in \mathfrak{A}(\mathfrak{h})$. We write $\rho_\Omega = |\Omega\rangle\langle\Omega|$ and $\langle\Phi\rangle = \langle\Omega|\Phi\Omega\rangle = \mathrm{tr}[\Phi\rho_\Omega]$ for operators Φ on $\mathscr{H}_\mathrm{E}$. By virtue of $\{\tau_t\}_{t\in\mathbb{R}}$ and π, the time evolution on $\mathscr{H}_\mathrm{E}$ is given by a strongly continuous unitary group $\left\{U_t^\mathrm{E}\right\}_{t\in\mathbb{R}}$ such that $\left[U_t^\mathrm{E}, \rho_\Omega\right] = 0$ for all $t \in \mathbb{R}$, i.e., its generator $\mathrm{i}H_\mathrm{E}$ satisfies $H_\mathrm{E}\Omega = 0$. Since ω is locally normal, $\mathscr{H}_\mathrm{E}$ is separable. The Hilbert space of the joint system is $\mathscr{H} = \mathscr{H}_\mathrm{S} \otimes \mathscr{H}_\mathrm{E}$, and the total free Hamiltonian is the essentially self-adjoint operator $H_0 = H_\mathrm{S} \otimes 1 + 1 \otimes H_\mathrm{E}$.

We now define a linear interaction between the system S and the environment E. It is given by the bounded self-adjoint Hamilton operator

$$H_\mathrm{I} = \sum_{k\in M} c_k(U^*(\sigma(k)) \otimes a^*(f_k) + U(\sigma(k)) \otimes a(f_k)). \tag{12}$$

Here U is a strongly continuous unitary representation of G on $\mathscr{H}_\mathrm{S}$ as in section 2, $\{f_k\}_{k\in\mathbb{Z}}$ is a sequence of test functions in $\mathfrak{h}$ to be defined below, $M \subseteq \mathbb{Z}$ is a subset such that $k \in M$ implies $-k \in M$ and $\sigma : M \longrightarrow G$ is a map such that $\sigma(-k) = (\sigma(k))^{-1}$ for all $k \in M$. The sequence $\{c_k\}_{k\in M} \subseteq \mathbb{R}_{\geqslant 0}$ with $c_k = c_{-k}$ ensures convergence of (12), we choose it such that $\sum_{k\in M} c_k < \infty$ and $\sum_{k\in M} c_k^2 < \infty$. Note that $U^*(\sigma(k)) = U(\sigma(-k))$. Finally, the creation and annihilation operators $a^*(f)$, $a(f)$ are those of the representation π, acting on $\mathscr{H}_\mathrm{E}$. Interaction (12) couples $U(\sigma(k))$ on $\mathscr{H}_\mathrm{S}$ with the creation/annihilation operator $a^\#(f_k)$. The total Hamiltonian of the joint system is the essentially self-adjoint operator $H_\mathrm{tot} = H_0 + H_\mathrm{I}$.

To complete the description of our model, we still have to choose the sequence of test functions $\{f_k\}_{k\in\mathbb{Z}}$. We assume that $\|f_k\| \leqslant 1$ for all $k \in \mathbb{Z}$, and we require that f_k and f_ℓ have disjoint energy spectra if $k \neq \ell$, that is ($\delta_{k,\ell}$ denotes the Kronecker symbol)

$$\omega(a_\mathrm{F}^*(f_k)\tau_t(a_\mathrm{F}(f_\ell))) = \langle a^*(f_k)a(\mathrm{e}^{\mathrm{i}tH} f_\ell)\rangle = \int_{\mathbb{R}^3} \overline{\hat{f}_\ell(p)}\hat{f}_k(p)\frac{\mathrm{e}^{-\beta p^2}}{1+\mathrm{e}^{-\beta p^2}}\,\mathrm{e}^{\mathrm{i}tp^2}\,\mathrm{d}p = \delta_{k,\ell}h_k(t), \tag{13}$$

where we have used (11). Then it follows that for all $k, \ell \in \mathbb{Z}$

$$\delta_{k,\ell}h_k(t) = \langle a^*(f_k)a(\mathrm{e}^{\mathrm{i}tH} f_\ell)\rangle, \qquad \delta_{k,\ell}\tilde{h}_k(t) = \langle a(f_k)a^*(\mathrm{e}^{\mathrm{i}tH} f_\ell)\rangle, \tag{14}$$

$$\delta_{k,\ell}\overline{h_k(t)} = \langle a^*(\mathrm{e}^{\mathrm{i}tH} f_k)a(f_\ell)\rangle, \qquad \delta_{k,\ell}\overline{\tilde{h}_k(t)} = \langle a(\mathrm{e}^{\mathrm{i}tH} f_k)a^*(f_\ell)\rangle, \tag{15}$$

where $h_k(t) = \left\langle a^*(f_k)a\left(\mathrm{e}_k^{\mathrm{i}tH}\right)\right\rangle$ and $\tilde{h}_k(t) = \langle a(f_k)a^*(\mathrm{e}^{\mathrm{i}tH} f_k)\rangle$. Note that we have $h_k(t) = \bar{h}_k(-t)$. Moreover, $h_k(t), \tilde{h}_k(t)$ are bounded functions, and $h_k(t), \tilde{h}_k(t) = \mathcal{O}(t^{-3})$ under suitable regularity assumptions, e.g., $f_k \in \mathscr{S}(\mathbb{R}^3)$. In the following, we will assume that $h_k = h_{-k}$ and $\tilde{h}_k = \tilde{h}_{-k}$ by a suitable choice of the f_k, e.g., $\hat{f}_k(p) = \hat{f}_{-k}(-p)$ for all $p \in \mathbb{R}^3$ and $\operatorname{supp} \hat{f}_k \cap \operatorname{supp} \hat{f}_\ell = \emptyset$ if $k \neq \ell$. This choice of $\{f_k\}_{k\in\mathbb{Z}}$, and of M and σ takes account of the isotropy of the scattering process.

We now make the connection to section 3. We take $\mathsf{X} = \mathsf{T}(\mathscr{H})$ and define the projection of norm 1 $P_0 = \iota \circ \mathrm{tr}_\mathrm{E}$, where $\iota : \mathsf{T}(\mathscr{H}_\mathrm{S}) \longrightarrow \mathsf{T}(\mathscr{H})$ with $\rho \mapsto \iota(\rho) = \rho \otimes \rho_\Omega$ is the inclusion of $\mathsf{T}(\mathscr{H}_\mathrm{S})$ in $\mathsf{T}(\mathscr{H})$ and $\mathrm{tr}_\mathrm{E} : \mathsf{T}(\mathscr{H}) \longrightarrow \mathsf{T}(\mathscr{H}_\mathrm{S})$ is the partial trace with respect to E. Then $\mathsf{X}_0 = P_0\mathsf{X} \cong \mathsf{T}(\mathscr{H}_\mathrm{S})$. We define the unbounded derivation $Z = -\mathrm{i}[H_0, \cdot]$ and the bounded derivation $A = -\mathrm{i}[H_\mathrm{I}, \cdot]$. Because $\left[U_t^\mathrm{E}, \rho_\Omega\right] = 0$, it follows that $[U_t, P_0] = 0$ for all $t \in \mathbb{R}$, remember that $U_t = \mathrm{e}^{Zt}$. Since $\langle a^\#(f)\rangle = 0$ for all $f \in \mathfrak{h}$ it follows that $A_{00} = 0$.

Next we calculate the explicit form of the operator K in (10). For all $\rho \in \mathsf{T}(\mathscr{H}_\mathrm{S})$, we have

$$K(\rho) = -\int_0^\infty P_0[H_\mathrm{I}, [H_\mathrm{I}(s), \rho \otimes \rho_\Omega]]\,\mathrm{d}s, \tag{16}$$

where $H_\mathrm{I}(s) = \mathrm{e}^{\mathrm{i}sH_\mathrm{E}} H_\mathrm{I}\,\mathrm{e}^{-\mathrm{i}sH_\mathrm{E}}$. Inserting (12) and remembering the definition of P_0 yields

$$\begin{aligned}
K(\rho) = -\int_0^\infty \sum_{k,\ell\in M} c_k c_\ell\{&+U(\sigma(k))U^*(\sigma(\ell))\rho\,\mathrm{tr}[a^*(f_k)a(\mathrm{e}^{\mathrm{i}tH} f_\ell)\rho_\Omega]\\
&+ U^*(\sigma(k))U(\sigma(\ell))\rho\,\mathrm{tr}[a(f_k)a^*(\mathrm{e}^{\mathrm{i}tH} f_\ell)\rho_\Omega]\\
&- U(\sigma(k))\rho U^*(\sigma(\ell))\,\mathrm{tr}[a^*(f_k)\rho_\Omega a(\mathrm{e}^{\mathrm{i}tH} f_\ell)]\\
&- U^*(\sigma(k))\rho U(\sigma(\ell))\,\mathrm{tr}[a(f_k)\rho_\Omega a^*(\mathrm{e}^{\mathrm{i}tH} f_\ell)]\\
&- U(\sigma(\ell))\rho U^*(\sigma(k))\,\mathrm{tr}[a^*(\mathrm{e}^{\mathrm{i}tH} f_\ell)\rho_\Omega a(f_k)]\\
&- U^*(\sigma(\ell))\rho U(\sigma(k))\,\mathrm{tr}[a(\mathrm{e}^{\mathrm{i}tH} f_\ell)\rho_\Omega a^*(f_k)]\\
&+ \rho U(\sigma(\ell))U^*(\sigma(k))\,\mathrm{tr}[\rho_\Omega a^*(\mathrm{e}^{\mathrm{i}tH} f_\ell)a(f_k)]\\
&+ \rho U^*(\sigma(\ell))U(\sigma(k))\,\mathrm{tr}[\rho_\Omega a(\mathrm{e}^{\mathrm{i}tH} f_\ell)a^*(f_k)]\}\,\mathrm{d}t
\end{aligned}$$

by noting that $\mathrm{tr}_\mathrm{E}[\rho_1 \otimes \Phi\rho_\Omega] = \rho_1\,\mathrm{tr}[\Phi\rho_\Omega]$ for $\rho_1 \in \mathsf{T}(\mathscr{H}_\mathrm{S})$ and $\Phi \in \mathscr{L}(\mathscr{H}_\mathrm{E})$. Using (14) and (15) we arrive at

$$K(\rho) = \sum_{k\in M} c_k^2 d_k(U^*(\sigma(k))\rho U(\sigma(k)) - \rho), \tag{17}$$

where

$$d_k = \int_0^\infty (h_{-k}(t) + \tilde{h}_k(t) + \overline{h_{-k}(t)} + \overline{\tilde{h}_k(t)})\,\mathrm{d}t, \qquad k \in M. \tag{18}$$

Note that $d_k \in \mathbb{R}$ and $d_k = d_{-k}$ for all $k \in M$. We assume that the sequence $\{d_k\}_{k\in M}$ is bounded (see below), in this case the series in (17) converges uniformly. Furthermore, for an integrable function f with $f(t) = \bar{f}(-t)$ we have

$$\int_0^\infty \mathrm{e}^{\mathrm{i}xt} f(t)\,\mathrm{d}t = \frac{1}{2}\hat{f}(x) + \mathrm{i}s(x), \qquad s(x) = \mathrm{Im}\int_0^\infty \mathrm{e}^{\mathrm{i}xt} f(t)\,\mathrm{d}t, \qquad x \in \mathbb{R}, \tag{19}$$

so that we can write $d_k = 2\,\mathrm{Re}\int_0^\infty h_{-k}(t)\,\mathrm{d}t + 2\,\mathrm{Re}\int_0^\infty \tilde{h}_k(t)\,\mathrm{d}t = \hat{h}_{-k}(0) + \hat{\tilde{h}}_k(0)$. Since h_k and $\tilde{h}_k$ are of positive type, i.e., $\sum_{i,j=1}^n \bar{z}_i z_j h_k(t_j - t_i) \geqslant 0$ for all $z_i \in \mathbb{C}, t_i \in \mathbb{R}, i = 1, \ldots, n, n \in \mathbb{N}$, it follows by Bochner's theorem that $\hat{h}_k(x), \hat{\tilde{h}}_k(x) \geqslant 0$ for all $x \in \mathbb{R}$, this shows that $d_k \geqslant 0$ for all $k \in M$. If we put now $n_k = c_k^2 d_k \geqslant 0$ for $k \in M$, we can write the generator of the semigroup $\{\exp((Z_0 + K)t)\}_{t\geqslant 0}$ in (9) as

$$L(\rho) = -\mathrm{i}[H_\mathrm{S}, \rho] + \sum_{k\in M} n_k (U^*(\sigma(k))\rho U(\sigma(k)) - \rho), \qquad \rho \in \mathrm{dom}[H_\mathrm{S}, \cdot], \tag{20}$$

with $n_k = n_{-k}$ for all $k \in M$. This generator determines the reduced Markovian dynamics of our discrete model, obtained in the singular coupling limit.

Our next task is the verification of the assumptions (7) and (8) of theorem 2 to obtain the convergence of the reduced dynamics to the semigroup generated by (20). To do so, we have to put additional constraints on the sequence of test functions $\{f_k\}_{k\in\mathbb{Z}}$.

Theorem 3. *Assume that there exists an integrable function h on* $\mathbb{R}$*, such that for all* $t_1, t_2 \in \mathbb{R}$ *and all* $k, \ell \in \mathbb{Z}$ *we have*

$$|\langle a^\#(\mathrm{e}^{\mathrm{i}t_1 H} f_k) a^\#(\mathrm{e}^{\mathrm{i}t_2 H} f_\ell)\rangle| \leqslant h(t_1 - t_2). \tag{21}$$

Moreover, assume that there exists $\varepsilon > 0$ *such that h satisfies*

$$\int_0^\infty |h(t)|(1+t)^\varepsilon \,\mathrm{d}t < \infty. \tag{22}$$

Then the assumptions (7) and (8) of theorem 2 are satisfied.

For a proof see [12]. The somewhat lengthy proof of theorem 3 follows—with minor changes since interaction (12) is linear in the field operators—similar lines as that in [11], so we do not reproduce it here. In the following, we will assume throughout that the test functions $\{f_k\}_{k\in\mathbb{Z}}$ satisfy the assumptions (21) and (22) of theorem 3.

Corollary 1. *With the notation and definitions as above it follows that*

$$\lim_{\lambda\to 0} P_0 U_t^\lambda P_0 \rho = \lim_{\lambda\to 0} \mathrm{tr}_\mathrm{E}[\exp((Z_0 + \lambda^{-2} Z_1 + \lambda^{-1} A)t)\rho \otimes \rho_\Omega] = \mathrm{e}^{Lt}\rho \tag{23}$$

uniformly on compact time intervals and for all $\rho \in \mathsf{T}(\mathscr{H}_\mathrm{S})$*, where L is given by (20).*

Finally we note the following simple fact, which will be used in the next section. A measure $\mu \in \mathscr{M}(G)$ is called discrete if it is of the form $\mu = \sum_{i=1}^n a_i \delta_{g_i}$, with $g_1, \ldots, g_n \in G$ and $a_1, \ldots, a_n \in \mathbb{R}$. Thus, we see that L in (20) can be written as $L_\mu(\rho) = -\mathrm{i}[H_\mathrm{S}, \rho] + \int_G (U(g)\rho U(g^{-1}) - \rho)\,\mathrm{d}\mu(g)$ with a discrete measure $\mu \in \mathscr{M}_+(G)$. By our choice of M, σ and the test functions this measure has the property that $\mu(B) = \mu(T(B))$ for all $B \in \mathscr{B}_0(G)$, which we denote by $\mu = \mu \circ T$, where $T : G \longrightarrow G$ is defined by $g \mapsto T(g) = g^{-1}$; as mentioned before, this property expresses the isotropy of the scattering process.

Corollary 2. *Let* $\mu \in \mathscr{M}_+(G)$ *be a discrete measure with* $\mu = \mu \circ T$*, i.e., of the form* $\mu = \sum_{k\in M} a_k \delta_{\sigma(k)}$ *with* $a_k \geqslant 0, a_k = a_{-k}$ *for all* $k \in M$*, where* $M \subseteq \mathbb{Z}$ *and σ are as above. Let* $H_\mathrm{I}(\mu)$ *be the Hamiltonian defined by (12), i.e.,* $H_\mathrm{I}(\mu) = \sum_{k\in M} c_k (U^*(\sigma(k)) \otimes a^*(f_k) + U(\sigma(k)) \otimes a(f_k))$ *with* $c_k = \sqrt{a_k/d_k}$*. Let* $\left\{P_0 U_t^\lambda(\mu) P_0\right\}_{t\in\mathbb{R}}$ *be the reduced dynamics with respect to the interaction Hamiltonian* $H_\mathrm{I}(\mu)$*, i.e., we have* $U_t^\lambda(\mu) = \exp((Z_0 + \lambda^{-2} Z_1 + \lambda^{-1} A(\mu))t)$ *and* $A(\mu) = -\mathrm{i}[H_\mathrm{I}(\mu), \cdot]$*. Then it follows that*

$$\lim_{\lambda\to 0} P_0 U_t^\lambda(\mu) P_0 \rho = \mathrm{e}^{L_\mu t}\rho, \tag{24}$$

uniformly on compact time intervals and for all $\rho \in \mathsf{T}(\mathscr{H}_\mathrm{S})$*, and with*

$$L_\mu(\rho) = -\mathrm{i}[H_\mathrm{S}, \rho] + \int_G (U(g)\rho U(g^{-1}) - \rho)\,\mathrm{d}\mu(g), \qquad \rho \in \mathrm{dom}[H_\mathrm{S}, \cdot]. \tag{25}$$

5. General case

We now generalize the considerations of the preceding section and show that models with a reduced dynamics described by (20), with the sum replaced by an integral with respect to a measure $\mu \in \mathscr{M}_+^{\mathrm{b}}(G)$ which satisfies $\mu = \mu \circ T$, can be approximated by the reduced dynamics of a Hamiltonian model with an interaction of the form (12). We start with two lemmas.

Lemma 1. *Let* X *be a Banach space and* $f : G \longrightarrow \mathsf{X}$ *be a bounded and norm continuous function. Assume that* $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathscr{M}_+^1(G)$ *is a sequence which converges to* $\mu \in \mathscr{M}_+^1(G)$ *in the vague topology. Then* $\left\| \int_G f \,\mathrm{d}\mu_n - \int_G f \,\mathrm{d}\mu \right\| \to 0$ *as* $n \to \infty$.

Proof. Let $\varepsilon > 0$. Since $\{\mu_n\}_{n\in\mathbb{N}} \cup \{\mu\}$ is compact in both vague and weak topology, and G is Hausdorff, it follows from théorème 2, chapter IX, section 5 of [15] that there is a compact set $K \subseteq G$ such that $\mu_n(G\backslash K), \mu(G\backslash K) < \varepsilon$ for all $n \in \mathbb{N}$. From the corollary of proposition 9, chapter III, section 3 of [15] it follows that $\mu \mapsto \int_K f \,\mathrm{d}\mu$ is continuous in the vague topology on vaguely bounded sets, hence $\left\| \int_K f \,\mathrm{d}\mu_n - \int_K f \,\mathrm{d}\mu \right\| < \varepsilon$ for n sufficiently large. Therefore, we have

$$\left\| \int_G f \,\mathrm{d}\mu_n - \int_G f \,\mathrm{d}\mu \right\| \leqslant \left\| \int_K f \,\mathrm{d}\mu_n - \int_K f \,\mathrm{d}\mu \right\| + \left\| \int_{G\backslash K} f \,\mathrm{d}\mu_n - \int_{G\backslash K} f \,\mathrm{d}\mu \right\| < \varepsilon(1 + 2\|f\|_\infty),$$

and the lemma follows. □

Lemma 2. *Let* $\mu \in \mathscr{M}_+^1(G)$ *with* $\mu = \mu \circ T$. *Then there is a sequence* $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathscr{M}_+^1(G)$ *of discrete measures with* $\mu_n = \mu_n \circ T$, *such that* $\mu_n \to \mu$ *as* $n \to \infty$ *in the vague topology.*

Proof. Since the discrete measures are dense in $\mathscr{M}_+^1(G)$ with respect to the vague topology, there is a sequence $\{\nu_n\}_{n\in\mathbb{N}}$ of discrete probability measures with $\nu_n \to \mu$ vaguely. Define $\mu_n = \frac{1}{2}(\nu_n + \nu_n \circ T)$. Then $\mu_n = \mu_n \circ T$ for all $n \in \mathbb{N}$, and $\mu_n \to \mu$ vaguely. □

Now let $\mu \in \mathscr{M}_+^1(G)$ with $\mu = \mu \circ T$ and $\alpha > 0$. We consider a Markovian time evolution on $\mathsf{T}(\mathscr{H}_\mathrm{S})$ generated by

$$L_\mu(\rho) = -\mathrm{i}[H_\mathrm{S}, \rho] + \alpha \int_G (U(g)\rho U(g^{-1}) - \rho)\,\mathrm{d}\mu(g), \qquad \rho \in \mathrm{dom}[H_\mathrm{S}, \cdot], \tag{26}$$

describing an isotropic scattering process. We show that the semigroup generated by (26) can be approximated by the reduced dynamics of a Hamiltonian model. According to lemma 2 there is a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathscr{M}_+^1(G)$ with $\mu_n = \mu_n \circ T$ such that $\mu_n \to \mu$ vaguely. For n fixed we consider the Hamiltonian $H_\mathrm{I}(\mu_n)$ defined in corollary 2. From (24) we have, for $\varepsilon > 0$ and sufficiently small λ, $\left\| P_0 U_t^\lambda(\mu_n) P_0 \rho - \mathrm{e}^{L_{\mu_n} t}\rho \right\|_1 < \varepsilon$ for all t in a compact interval. Next we note that $g \mapsto V_g(\rho) := U(g)\rho U(g^{-1})$ is norm continuous since V is a weakly continuous representation of G on the Banach space $\mathsf{T}(\mathscr{H}_\mathrm{S})$. Therefore, it follows from lemma 1 that $L_{\mu_n}(\rho) \to L_\mu(\rho)$ as $n \to \infty$ for all $\rho \in \mathrm{dom}\, L_\mu = \mathrm{dom}\, L_{\mu_n} = \mathrm{dom}[H_\mathrm{S}, \cdot]$, which is dense in $\mathsf{T}(\mathscr{H}_\mathrm{S})$, L_μ is given by (26). The Trotter–Kato theorem now implies $\mathrm{e}^{L_{\mu_n} t}\rho \to \mathrm{e}^{L_\mu t}\rho$ for all $\rho \in \mathsf{T}(\mathscr{H}_\mathrm{S})$ uniformly for t in compact intervals. Thus, $\|\mathrm{e}^{L_{\mu_n} t}\rho - \mathrm{e}^{L_\mu t}\rho\|_1 < \varepsilon$ for n sufficiently large, and we have

$$\left\| P_0 U_t^\lambda(\mu_n) P_0 \rho - \mathrm{e}^{L_\mu t}\rho \right\|_1 \leqslant \left\| P_0 U_t^\lambda(\mu_n) P_0 \rho - \mathrm{e}^{L_{\mu_n} t}\rho \right\|_1 + \|\mathrm{e}^{L_{\mu_n} t}\rho - \mathrm{e}^{L_\mu t}\rho\|_1 < 2\varepsilon \tag{27}$$

for all t in a compact interval, n sufficiently large and λ sufficiently small. This proves the following theorem.

Theorem 4. *Let* $\mu \in \mathscr{M}_+^1(G)$ *be a measure with* $\mu = \mu \circ T$*, and consider the generator* L_μ *given by (26), describing an isotropic scattering process. Then for every* $t_0 \geqslant 0$, $\rho \in \mathsf{T}(\mathscr{H}_\mathrm{S})$ *and* $\varepsilon > 0$*, there exists a Hamiltonian* H_I *of the form (12), and* $\lambda_0 > 0$ *such that for the reduced dynamics* $\{P_0 U_t^\lambda P_0\}_{i\in\mathbb{R}}$ *with respect to the interaction Hamiltonian* H_I*, i.e., with* $U_t^\lambda = \mathrm{e}((Z_0 + \lambda^{-2} Z_1 + \lambda^{-1} A)t)$, $A = -\mathrm{i}[H_\mathrm{I}, \cdot]$*, we have*

$$\left\| P_0 U_t^\lambda P_0 \rho - \mathrm{e}^{L_\mu t} \rho \right\|_1 < \varepsilon, \qquad \text{for all } t \in [0, t_0] \tag{28}$$

if $0 < \lambda < \lambda_0$.

In particular, this theorem shows that the semigroup of Alicki's model, generated by (4), can be approximated by the reduced dynamics of a Hamiltonian interaction between the system S and environment E, provided the scattering process is isotropic, i.e., $n(k) = n(|k|)$ for all $k \in \mathbb{R}^3$. In this case it follows that $\mu = n\lambda$ satisfies $\mu = \mu \circ T$, and theorem 4 applies.

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References

[1] Joos E, Zeh H D, Kiefer C, Giulini D, Kupsch J and Stamatescu I-O 2003 *Decoherence and the Appearance of a Classical World in Quantum Theory* 2nd edn (Berlin: Springer)

[2] Blanchard Ph, Giulini D, Joos E, Kiefer C and Stamatescu I-O (ed) 2000 *Decoherence: Theoretical, Experimental, and Conceptual Problems: Proceedings (Bielefeld 1998)* (Berlin: Springer)

[3] Hornberger K, Uttenthaler S, Brezger B, Hackermüller L, Arndt M and Zeilinger A 2003 Collisional decoherence observed in matter wave interferometry *Phys. Rev. Lett.* **90** 160401

[4] Joos E and Zeh H D 1985 The emergence of classical properties through interaction with the environment *Z. Phys.* B **59** 223–43

[5] Gallis M R and Fleming G N 1990 Environmental and spontaneous localization. *Phys. Rev.* A **42** 38–48

[6] Hornberger K and Sipe J E 2003 Collisional decoherence reexamined *Phys. Rev.* A **68** 012105

[7] Alicki R and Lendi K 1987 *Quantum Dynamical Semigroups and Applications (Lecture Notes in Physics* vol 286*)* (Berlin: Springer)

[8] Alicki R 2002 Search for a border between classical and quantum worlds *Phys. Rev.* A **65** 034104

[9] Kossakowski A 1972 On quantum statistical mechanics of non-Hamiltonian systems *Rep. Math. Phys.* **3** 247–78

[10] Palmer P F 1974 The singular and weak coupling limits *J. Math. Phys.* **18** 527–9

[11] Davies E B 1974 Markovian master equations *Commun. Math. Phys.* **39** 91–110

[12] Hellmich M 2004 *Diploma Thesis* University of Bielefeld

[13] Davies E B 1980 *One-Parameter Semigroups* (London: Academic)

[14] Bratteli O and Robinson D W 1987/1981 *Operator Algebras and Quantum Statistical Mechanics* 2nd edn, vol 1 and 2 (New York: Springer)

[15] Bourbaki N 1959–1969 *Éléments de mathématique Livre VI: Intégration* chapters 1–9 (Paris: Hermann)